Mathematica Moravica
Vol. **11** (2007), 69–78

L-Ultrafilters, L-Sets and Lc-Property

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Abstract. An L-filter base, L-filter, L-ultrafilter is a filter base, filter, ultrafilter consisting exclusively of Lindelöf sets. In this paper we consider L-filters (ultrafilters) and LC-property. A space X is $LC-space$ if every Lindelöf set in X has the compact closure in X. A locally compact space X is $LC-space$ if and only if every L-ultrafilter on X converges. We also consider L-points, L-sets and LC-extensions.

1. Introduction and definitions

Throughout this paper, all spaces are assumed to be regular Hausdorff. The closure of a subset A of a space X is denoted by $cl_X(A)$. We use the standard definitions for filter-base and filter.

For a space X, let $\mathfrak{K}(X)$ and $\mathfrak{F}(X)$ denote the families of all nonempty compact subsets and finite subsets of X, respectively. We assume that$\mathfrak{K}(X)$ has the finite topology and $\mathfrak{F}(X)$ is a subspace of $\mathfrak{K}(X)$. That is, $\mathfrak{K}(X)$ has a base consisting of the sets of the form:

$$\langle U_1, \ldots, U_k \rangle = \{K \in \mathfrak{K}(X) : K \subset \cup_{i=1}^{k} U_i \wedge K \cap U_i \neq \emptyset \text{ for each } i\},$$

where $\{U_1, \ldots, U_k\}$ is a finite family of open subsets of X.These spaces are studied in [6]. The symbol $\mathfrak{L}(X)$ denote the family of all nonempty Lindelöf subsets of X. $\mathfrak{L}(x)$ denotes the set of all Lindelöf neighbourhoods of $x \in X$.

A topological space X is called a locally Lindelöf space if for every $x \in X$ there exists a neighbourhood U of the point x such that $cl_X(U)$ is a Lindelöf subspace of X.

It is easy to see that every locally compact space is a locally Lindelöf space.

Definition 1.1. A Hausdorff space X is called absolutely closed (or H-closed) if X is closed in every Hausdorff space in which is embedded ([3]).

Lemma 1.1. *A regular space X is compact if and only if every open ultrafilter on X converges* ([3]).

2000 *Mathematics Subject Classification.* Primary: 54D30.
Key words and phrases. L-set, LC-space, LC-extensions.

2. L-filters

A filter-base in $\mathfrak{L}(X)$ is a non-empty family $\mathfrak{B} \subset \mathfrak{L}(X)$ such that if $A_1,\ A_2 \in \mathfrak{B}$, then there exists an $A_3 \in \mathfrak{B}$ such that $A_3 \subset A_1 \cap A_2$.

Definition 2.1. A L-filter is a nonempty subfamily $\mathfrak{F} \subset \mathfrak{L}(X)$ satisfying the following conditions:

(a) $\emptyset \ni \mathfrak{F}$.

(b) If $A_1, A_2 \in \mathfrak{F}$, then $A_1 \cap A_2 \in \mathfrak{F}$.

(c) If $A \in \mathfrak{F}$ and $G \in \mathfrak{L}(X)$ such that $A \subset G$, then $G \in \mathfrak{F}$.

A filter $\mathfrak{U}$ in $\mathfrak{L}(X)$ is a maximal filter or a $L - ultrafilter$, if for every filter $\mathfrak{F}$ in $\mathfrak{L}(X)$ that contains $\mathfrak{U}$ we have $\mathfrak{F} = \mathfrak{U}$.

One readily sees that for any filter-base $\mathfrak{B}$ in $\mathfrak{L}(X)$, the family $\mathfrak{F}_{\mathfrak{B}} = \{A \in \mathfrak{L}(X) : \textit{there exists a } B \in \mathfrak{B} \textit{ such that} B \subset A\}$ is a filter in $\mathfrak{L}(X)$.

Definition 2.2. Let X be a locally Lindelöf space.

(a) A point $x \in X$ is called a *limit* of a L-filter $\mathfrak{F}$ if $\mathfrak{L}(x) \subset \mathfrak{F}$; we then say that the L-filter $\mathfrak{F}$ converges to x and write $x \in lim\mathfrak{F}$.

(b) A point x is called a *limit* of a filter base $\mathfrak{B} \subset \mathfrak{L}(X)$ if $x \in lim\mathfrak{F}_{\mathfrak{B}}$; we then say that the filter base $\mathfrak{B}$ converges to x and write $x \in lim\mathfrak{B}$.

Remark 2.1. Clearly, $x \in lim\mathfrak{B}$ if and only if every compact neighbourhood of x contains a member of $\mathfrak{B}$.

Definition 2.3. Let X be a locally Lindelöf space. A point x is called a *cluster point* of a L-filter $\mathfrak{F}$ (or a filter base $\mathfrak{B}$) if x belongs to the closure of every member of $\mathfrak{F}$ (of $\mathfrak{B}$).

Remark 2.2. Clearly, x is cluster point of a L-filter $\mathfrak{F}$ (or a filter base $\mathfrak{B}$) if and only if every compact neighborhood of x intersects all members of $\mathfrak{F}$(or $\mathfrak{B}$). This implies, in particular, that every cluster point of a L-ultrafilter is a limit of this ultrafilter.

Lemma 2.1. *If $\mathfrak{U}$ is a L-ultrafilter in $\mathfrak{L}(X)$, the following holds:*

(a) *If $A \in \mathfrak{L}(X)$, then $A \cap U \neq \emptyset$ for all $U \in \mathfrak{U}$ iff $A \in \mathfrak{U}$*

(b) *If A_1, A_2 are Lindelöf subsets of X and $A_1 \cup A_2 \in \mathfrak{U}$, then $A_1 \in \mathfrak{U}$ or $A_2 \in \mathfrak{U}$.*

Proof. $(a) \Leftarrow$: If $A \in \mathfrak{U}$, then $A \cap U \neq \emptyset$ for all $U \in \mathfrak{U}$.

$\Rightarrow$: If $A \cap U \neq \emptyset$ for all $U \in \mathfrak{U}$ and $A \notin \mathfrak{U}$, then $\mathfrak{U} \cup \{A\}$ is a filter base in $\mathfrak{L}(X)$, that contains $\mathfrak{U}$. Since $\mathfrak{U}$ is a L-ultrafilter in $\mathfrak{L}(X)$, it follows that $A \in \mathfrak{U}$.

(b): Suppose that $A_1 \notin \mathfrak{U}$, $A_2 \notin \mathfrak{U}$ and $A_1 \cup A_2 \in \mathfrak{U}$. Let $\mathfrak{B}$ be a subfamily of $\mathfrak{L}(X)$. The set $A \in \mathfrak{L}(X)$ is a member of $\mathfrak{B}$ iff $A \cup A_1 \in \mathfrak{U}$. Clearly, $\mathfrak{B}$ is a L-filter that contains $\mathfrak{U}$. Since $\mathfrak{U}$ is a L-ultrafilter in $\mathfrak{L}(X)$, it follows that $A_1 \in \mathfrak{U}$ or $A_2 \in \mathfrak{U}$. $\diamondsuit$

Lemma 2.2. *Let X be a hereditarily Lindelöf (closed, σ -compact, F_σ) subset of a topological space Y and let $\mathfrak{F}$ be a L - filter in $\mathfrak{L}(Y)$. The family $\mathfrak{F}_X = \mathfrak{F} \cap X = \{F \cap X : F \in \mathfrak{F}\}$ is a L - filter in $\mathfrak{L}(X)$ if and only if $F \cap X \neq \emptyset$, for every $F \in \mathfrak{F}$.*

Proof. (a) Empty set $\emptyset \notin \mathfrak{F}_X \iff F \cap X \neq \emptyset$ for all $F \in \mathfrak{F}$. Furthermore, every member of $\mathfrak{F}_X$ is a Lindelöf subset of X.

(b) Let sets $A_1 \cap X$ and $A_2 \cap X$ be contained in $\mathfrak{F}_X$. Then $(A_1 \cap X) \cap (A_2 \cap X) = (A_1 \cap A_2) \cap X \in \mathfrak{F}_X$, $(A_1 \cap A_2 \in \mathfrak{F})$.

(c) Also, if $A \cap X \in \mathfrak{F}$ and B is a Lindelöf subset in $\mathfrak{L}(X)$, $A \subset B$; then $A \cup B \in \mathfrak{L}(Y)$ and $A \cup B \in \mathfrak{F}$. We have $B = (A \cup B) \cap X \in \mathfrak{F}_X$. So, we have shown that $\mathfrak{F}_X$ is a L - filter on X. $\diamondsuit$

The following is an immediate consequence of Lemmas 2.2. and 2.1.

Lemma 2.3. . *Let X be a hereditarily Lindelöf (σ- compact, F_σ) subset of a topological space Y and let $\mathfrak{F}$ be a L - ultrafilter on Y. The family $\mathfrak{F}_X = \mathfrak{F} \cap X = \{F \cap X : F \in \mathfrak{F}\}$ is a L - ultrafilter in $\mathfrak{L}(X)$ if and only if $X \in \mathfrak{F}$.*

Lemma 2.4. *.Let X be a hereditarily Lindelöf (σ - compact, F_σ) subspace of a locally compact space Y. If every L - ultrafilter on Y converges, then every L - ultrafilter on X converges to some point in $cl_Y(X)$.*

Proof. Let $\mathfrak{U}$ be a L - ultrafilter on X. Since the subset $X \subset Y$ is σ - compact, it is easy to see that $X \in \mathfrak{U}$. It is clear that family $\mathfrak{U}$ is a L - filter base on Y. Let $\mathfrak{U}'$ be the L - ultrafilter on Y generated by $\mathfrak{U}$. Now suppose $\mathfrak{U}' \to p \in Y$. By Definition 2.3., $p \in \lim \mathfrak{U}' \iff p \in cl_Y(U')$ for each $U' \in \mathfrak{U}'$. Since the family $\mathfrak{U} \subset \mathfrak{U}'$, the point $p \in cl_Y(U)$ for each $U \in \mathfrak{U}$. Hence $p \in \lim \mathfrak{U}$. $\diamondsuit$

Proposition 2.5. *Let X be a Lindelöf, dense subspace of a locally compact space Y. The space Y is compact if and only if every L - ultrafilter on X converges to some point in Y.*

Proof. Let Y be a compact space. It is known that every ultrafilter on Y converges; in particular, every L - ultrafilter on Y converges. From Lemma 2.4., it follows that every L - ultrafilter on X converges to some point in Y. Conversely, suppose that every L - ultrafilter on X converges. We shall prove that every open ultrafilter on Y converges. Since Y is locally compact and Hausdorff it is Tychonoff. By Lemma 1.1., Y is a compact space. If $\mathfrak{U}'$ is an open ultrafilter on Y and $\mathfrak{U} = \mathfrak{U}' \cap X = \{U' \cap X : U' \in \mathfrak{U}'\}$, then $\mathfrak{U}$ is an open filter on X. Clearly the family $\mathfrak{B} = \{cl_Y(U') \cap X : U' \in \mathfrak{U}'\}$ is a filter base in $\mathfrak{L}(X)$ (L - filter base on X). Let $\mathfrak{F}$ be the L - ultrafilter on X generated by $\mathfrak{B}$. Now suppose that $\mathfrak{F} \to p \in Y = cl_Y(X)$. From Definition 2.3., it follows that $p \in \lim \mathfrak{F} \iff p \in cl_Y(cl_Y(U') \cap X)$ for each $U' \in \mathfrak{U}'$. Therefore, for each $U' \in \mathfrak{U}$, we have that $p \in cl_Y(U') \cap cl_Y(X) = cl_Y(U') \cap Y$. Since, $p \in cl_Y(U')$, for each $U' \in \mathfrak{U}'$, it is clear that $\mathfrak{U}'$ converges to p. $\diamondsuit$

3. L-sets and LC-property

Now, we introduce the definitions of L-point, L-set and LC-property, which will be used to be useful in the later discussion.

Definition 3.1. Let X be a topological space.

(a) A point $p \in X$ is an $L - point$ if $p \notin cl_X(F)$ for each Lindelöf subset $F \subset X \setminus \{p\}$.

(b) A set $A \subset X$ is an $L - set$ if $A \cap cl_X(F) = \emptyset$ for each Lindelöf subset F contained in $X \setminus A$.

The symbol $\mathfrak{L}_X$ denotes the set of all L-point of X. It is easy to see that set $\mathfrak{L}_X$ is a L-set. The converse is not necessarily true. Let R be the set of real numbers. We define the Euclidean topology on the set R by using the basis sets of the form $(a, b) = \{x \in R : a < x < b\}$. It is known that R is second countable, separable, locally compact and σ-compact.

Every open interval (a, b) is a L-set, since for every Lindelöf subset $A \subset (-\infty, a] \cup [b, +\infty)$ the closure $cl_R(A) \subset (-\infty, a] \cup [b, +\infty)$. But $\mathfrak{L}_X = \emptyset$, since R is second countable.

Definition 3.2. Let X be a topological space.

(a) A point $p \in X$ is said to be a $P - point$ if the intersection of countably many neighborhoods of p is a neighborhood of p .

(b) A point $p \in X$ is a $weakP - point$ if $p \notin cl_X(F)$ for each countable subset $F \subset X \setminus \{p\}$ [9].

It is clear that every P-point is a weak P-point. Furthermore, it can be shown that a point $p \in X$ is a P-point if and only if every F_σ-set that is contained in $X \setminus \{p\}$ has the closure contained in $X \setminus \{p\}$.

Definition 3.3. Let X be a topological space.

(a) A set $A \subset X$ is said to be a P-set if the intersection of countably many neighborhoods of A is a neighborhood of A.

(b) A set $A \subset X$ is a weak P-set($wP-set$) if $A \cap cl_X(F) = \emptyset$ for each countable set F contained in $X \setminus A$ [9].

It is easy to see that every P-set is a weak P-set and every open set of X is a P-set.

The reader can easily prove the following lemma.

Lemma 3.1. *Let X be a topological space. The set $A \subset X$ is a P-set if and only if every F_σ-set that is contained in $X \setminus A$ has the closure contained in $X \setminus A$. If X is a compact space, then the set $A \subset X$ is a P-set if and only if every σ-compact set that is contained in $X \setminus A$ has the compact closure contained in $X \setminus A$.*

Lemma 3.2. *Let X be a regular space. Then every closed P-set is an L-set.*

Proof. Let A be the closed P-set in X and let L be a Lindelöf subset of $X \setminus A$. Since the space X is regular, for every point $x \in L$ there exist an open set $O_x \subset X \setminus A$ such that $cl_X(O_x) \subset X \setminus A$. The family $\{O_x : x \in L\}$ is an open cover of L. Since L is Limdelöf, there exist countably subfamily $\mathfrak{U} = \{O_n : n \in N\} \subset \{O_x : x \in L\}$ such that $L \subset \cup\{cl_X(O_n); n \in N\} \subset X \setminus A$. The set $\cup\{cl_X(O_n); n \in N\}$ is an F_σ subset of $X \setminus A$ such that $L \subset \{cl_X(O_n) : n \in N\}$ and $\{cl_X(O_n) : n \in N\} \subset X \setminus A$. By definition 3.2., is clear that $cl_X(L) \subset cl_X(\cup\{cl_X(O_n); n \in N\}) \subset X \setminus A$. $\diamondsuit$

Corollary 3.3. *Let X be a regular T_1 space. Then every P - point is an L- point.*

It is clear that every L-point is a weak P-point. Furthermore, the set $\mathfrak{L}_X$ is a L-set. The following example shows that no every weak P - point is an L - point.

Example 3.1. Let $[0, \omega_1]$ ($[0, \omega_0]$) be the space of ordinals less than or equal to the first uncountable ordinal (first countable ordinal) with the order topology and $[0, \omega_1] \times [0, \omega_0]$ the Cartesian product. The subspace $X_1 = [0, \omega_1] \times [0, \omega_0] \setminus \{(\omega_1, n) : n \in [0, \omega_0)\}$ of $[0, \omega_1] \times [0, \omega_0]$ is noncompact and normal in the subspace topology. Let $X_2 = X_1 \cup \{p\}$, ($p \notin X_1$) be the one-point compactification of X_1. Then the space X_2 is compact and T_1 space. It is not Hausdorff since the point p and (ω_1, ω_0) have no disjoint neighbourhoods. The point (ω_1, ω_0) is a weak P - point but it is not a P - point.

Let $A = \{a_n \in X_2 : n \in N\}$ be any countable subset of $X_2 \setminus \{(\omega_1, \omega_0)\}$ and let $p \in A$. Then $A = \{(x_n, y_n) : x_n \in [0, \omega_1),\ y \in [0, \omega_0); n \in N\}$ where $\{x_n \in [0, \omega_1) : n \in N\} \subset [0, \omega_1)$ and $\{y_n \in [0, \omega_0) : n \in N\} \subset [0, \omega_0)$. Let a be an upper bound for the x_n; $a < \omega_1$, since ω_1 has uncountably many predecessors, while a has only countably many. Thus the set $([0, a] \times [0, \omega_0]) \cup \{p\}$ is closed and compact in X_2.

Furthermore, $A \subseteq ([0,a]\times[0,\omega_0])\cup\{p\}$ and $(\omega_1,\omega_0) \notin ([0,a]\times[0,\omega_0])\cup\{p\}$. Then $(\omega_1,\omega_0) \notin ([0,a]\times[0,\omega_0])\cup\{p\}$.

The point (ω_1,ω_0) is not an L - point because there exists a σ - compact (Lindelöf) subset $F = \cup\{([0,\omega_1)\times\{k\})\cup\{p\} : k \in [0,\omega_0)\} \subset X_2$ such that $cl_{X_2}(F) = X_2$.

We need now the following simple lemma taken from 3.1.

Lemma 3.4. *Let X be a compact space. The set (point) $A \subset X (a \in X)$ is a L-set(point) if and only if every Lindelöf set that is contained in $X \setminus A$ $(X\setminus\{a\})$ has the compact closure contained in $X\setminus A$ $(X\setminus\{a\})$.*

The following example shows that not every P - point is an L - point.

Example 3.2. Let $[0,\omega_1]$ $([0,\omega_0])$ be the space of ordinals less than or equal to the first uncountable ordinal (first countable ordinal) with the order topology and $[0,\omega_1]\times[0,\omega_0]$ Cartesian product. The subspace $X_1 = [0,\omega_1]\times[0,\omega_0]\setminus\{(\omega_1,n) : n\in[0,\omega_0)\}$ of $[0,\omega_1]\times[0,\omega_0]$ is noncompact and normal in the subspace topology. Let $X_2 = X_1\cup\{p\}$, $(p \notin X_1)$ be the one-point extension of X_1. We can define an topology on X_2 by declaring open base of the point p any subset of X_2 whose complement is Lindelöf subset. Then the space X_2 is Lindelöf and T_1 space. It is not Hausdorff (compact) since the point p and (ω_1,ω_0) have no disjoint neighbourhoods (since the subsets $([0,\omega_1)\times\{n\})\cup\{p\}; n\in[0,\omega_0)$ are closed and noncompact subsets in X_2).

The point (ω_1,ω_0) is a P - point but not an L - point. Let $\mathfrak{A} = \{A_n \subset X_2 : n\in N\}$ be any σ - compact subset of $X_2\setminus(\omega_1,\omega_0)$.

Case I: If $\{p\}\notin\mathfrak{A}$ then $p_1(\mathfrak{A})$ and $p_0(\mathfrak{A})$ are σ - compact subsets of $[0,\omega_1)$ and $[0,\omega_0]$, where p_1,p_0 are projections from $X_2\setminus\{(\omega_1,\omega_0)\}$ onto $[0,\omega_1)$, $[0,\omega_0]$. Since $[0,\omega_1)([0,\omega_0])$ is hypercountably [1] compact (compact) there exists a compact subsets $[0,\alpha]\subset[0,\omega_1]$ and $[0,\omega_0]$ such that $p_1(\mathfrak{A})\subset[0,\alpha]$ and $p_0(\mathfrak{A})\subset[0,\omega_0]$. The set $[0,\alpha]\times[0,\omega_0]$ is closed and compact in $X_2\setminus\{(\omega_1,\omega_0)\}$. Furthermore, $\mathfrak{A}\subset[0,\alpha]\times[0,\omega_0]$ and $(\omega_1,\omega_0)\notin cl_{X_2}(\mathfrak{A})$.

Case II : Let $\{p\}\in\mathfrak{A}$. We will now show that (ω_1,ω_0) is a P - point. According to Case I, the set $\mathfrak{A}\subset([0,\alpha]\times[0,\omega_0])\cup\{p\}$. Since $([0,\alpha]\times[0,\omega_0])\cup\{p\}$ is closed and compact in $X_2\setminus\{(\omega_1,\omega_0)\}$, the point (ω_1,ω_0) is a P - point in X_2.

The point (ω_1,ω_0) is not an$L$ - point because there exists a Lindelöf subset $F = \cup\{([0,\omega_1)\times\{k\})\cup\{p\} : k\in[0,\omega_0)\}\subset X_2$ such that $cl_{X_2}(F) = X_2$.

Remark 3.1. The subspace $X_2\setminus\{(\omega_1,\omega_0)\}$ in Example 3.2., is a hypercountably compact (HCC) space but it is not an LC - space.

[1]We shall say that the space X is *hypercountably compact*(*strongly countably compact*) if every σ-compact (countable) subset of X has a compact closure see[8]

Definition 3.4. A topological space X will be called an $LC - space$ if each Lindelöf subspace of X has compact closure.

The following is an immediate consequence of Lemma 3.4., and Definition 3.4.

Lemma 3.5. *A Tychonoff space X is $LC - space$ if and only if for every compactification cX of the space X the remainder $cX \setminus c(X)$ is an L - set in cX.*

Theorem 3.6. *For every Tychonoff space X the following conditions are equivalent:*

(I) For every compactification cX of the space X the remainder $cX \setminus c(X)$ is an L - set in cX.

(II) The remainder $\beta X \setminus \beta(X)$ is an L - set in βX.

(III) There exists a compactification cX of the space X the remainder $cX \setminus c(X)$ is an L - set in cX.

Proof. Implications $(I) \Rightarrow (II)$ and $(II) \Rightarrow (III)$ are obvious, so that it suffices to prove that $(III) \Rightarrow (I)$.

$(III) \Rightarrow (I) case : (III) \Rightarrow LC$ property and by Lemma 3.5., LC property $\Leftrightarrow (I).\diamondsuit$

A topological space X is $L - complete$ if X is a Tychonoff space and satisfies condition (I), and hence all the conditions, in Theorem 3.6.

Theorem 3.7. *For every Hausdorff locally compact (Tychonoff) space X the following conditions are equivalent:*

(I) The space X is LC.

(II) For every compactification cX the remainder $cX \setminus c(X)$ is a L-set in cX.

(III) The remainder $\beta X \setminus \beta(X)$ is an L - set in βX.

(IV) There exists a compactification cX of the space X such that the remainder $cX \setminus c(X)$ is a L-set in cX.

(V) Every L - filter base on X has a cluster point.

(VI) Every L - ultrafilter on X converges.

Proof. $(I) \Rightarrow (II)$. Since X is locally compact for every compactification cX the remainder $cX \setminus c(X)$ is a closed (compact) set in cX. If X is LC space, then every Lindelöf set in $c(X)$ has the compact closure in $c(X)$. By Lemma 3.5, $cX \setminus c(X)$ is a P-set.

By Lemma 3.5., and Theorem 3.6., $(I) \Leftrightarrow (II) \Leftrightarrow (III) \Leftrightarrow (IV)$.

$(V) \iff (VI)$. Obvious.

$(I) \Rightarrow (VI)$. Let $\mathfrak{U}$ be a L - ultrafilter on X and U_0 be a member of $\mathfrak{U}$. Since X is LC, $cl_X(U_0) = Y$ is a compact subspace of X. Let $\mathfrak{U}'$ be the trace of $\mathfrak{U}$ on Y. By Lemma 2.2., $\mathfrak{U}'$ is a L - filter on Y. Since Y is compact, there exists a point

$p \in Y$ such that p is a cluster point of $\mathfrak{U}'$. Clearly the point p is a cluster point of $\mathfrak{U}$. Hence every L - ultrafilter on X converges.

$(VI) \Rightarrow (I)$. Suppose that every L - ultrafilter on X converges and let A be any Lindelóf subset of X. We shall prove that $B = cl_X(A)$ is compact subset of X. Clearly, B is Tychonoff. If every open ultrafilter on B converges, By Lemma 1.1, B is compact. Let $\mathfrak{U}'$ be a open ultrafilter on B and $\mathfrak{U} = \mathfrak{U}' \cap A = \{U' \cap A : U' \in \mathfrak{U}'\}$. It is easy to see that the family $\mathfrak{B} = \{cl_X(U') \cap A : U' \in \mathfrak{U}'\}$ is a filter base in $\mathfrak{L}(X)$. Let $\mathfrak{F}$ be the L-ultrafilter on X generated by $\mathfrak{B}$. Now suppose that $\mathfrak{F} \longrightarrow p \in X$. From Definition 2.3., it follows that $p \in lim\mathfrak{F} \iff p \in cl_X(cl_X(U') \cap A)$ for each $U' \in \mathfrak{U}'$. Therefore, for each $U' \in \mathfrak{U}'$, we have that $p \in cl_X(U') \cap cl_X(A) = cl_X(U') \cap B$. Since, $p \in cl_X(U') = cl_B(U')$, for each $U' \in \mathfrak{U}'$, it is clear that $\mathfrak{U}'$ converges to $p \in B$, Hence, X is a LC-space.$\diamondsuit$

Proposition 3.8. *Every closed subspace of an LC - space is LC .*

Proof. Since Lindelöfness is hereditary with respect to closed subsets, it immediately follows from Lemma 3.5. $\diamondsuit$

Since compactness and Lindelöfness is hereditary with respect to closed subsets and finite unions, the following proposition is a consequence of Lemma 3.5 and Proposition 3.8.

Proposition 3.9. *The sum $\oplus\{X_s : s \in S\}$ is LC - space if and only if all spaces X_s are LC and the set S is finite.*

Proposition 3.10. *The Cartesian product of LC - spaces is LC .*

Proof. Let $X = \times\{X_a : a \in A\}$ be the product of LC - spaces X_a and let F be any Lindelöf subset of X. Since the projections $p_a : X \longrightarrow X_a$ from X onto X_a are continuous and open mappings, from each $a \in A$, we have that $p_a(F)$ is a Lindelöf subset of X_a. The set $cl_{X_a}(p_a(F))$ is compact in X_a. Furthermore, $F \subset Y = \times\{cl_{X_a}(p_a(F)) : a \in A\}$ and Y is a compact(closed) subspace of X. Then $cl_X(F) = cl_Y(F)$ is a compact subset of X. By Theorem 3.6. and Lemma 3.5., X is an LC - space.$\diamondsuit$

Corollary 3.11. *The limit of an inverse sequence of LC - spaces is LC .*

Proposition 3.12. *Let X be the product of spaces X_i, $i \in \{1, 2 \ldots, n\}$. If X is LC - complete space, then every X_i are LC .*

Proof. Let F_j be any Lindelöf subset of $X_j, j \in \{1,\ 2 \ldots, n\}$. For a fixed $x \in \times\{X_i : i \in \{1,\ 2 \ldots, n\} \setminus \{j\}\}$ the set $A = F \times \{x\} \subset X$ is a Lindelöf subset of X. Since X is an LC - space, $cl_X(A) \in \mathfrak{K}(X)$. For each $X_j\ : j \in \{1,\ 2, \ldots, n\}$, we have that $p_j(cl_X(A)) \in \mathfrak{K}(X_j)$, where p_j is the projection from X onto X_j.

Furthermore, $F \subset p_j(cl_X(A))$ and $cl_{X_j}(F) \in \mathfrak{K}(X_j)$. Hence, X_j is a LC - space.$\diamondsuit$

Since the class of compact (Lindelöf) spaces is perfect, from the definition of LC - spaces we obtain.

Proposition 3.13. *If X and Y are Tychonoff spaces and there exists a perfect mapping $f : X \longrightarrow Y$ of X onto Y, then X is L - space if and only if Y is LC - space.*

Lemma 3.14. *Let $\mathfrak{L}$ be a Lindelöf subset of $\mathfrak{K}(X)$. Then the set $\|\mathfrak{L}\| = \cup\{L \in \mathfrak{L}\}$ is a Lindelöf subset of X.*

Proof. Suppose $A = \|\mathfrak{L}\| = \cup\{L \in \mathfrak{L}\}$, with $\mathfrak{L}$ is a Lindelöf subset of $\mathfrak{K}(X)$. Let $\mathfrak{U}$ be a Collection of open subsets of X which covers A. Now let $L \in \mathfrak{L}$; then L is a compact subset of X, and hence there exists a finite subcollection $\{U_{L_1}, \dots U_{L_n}\}$ of $\mathfrak{U}$ which covers L, and all of whose elements intersect L. Hence, for each $L \in \mathfrak{L}, \mathfrak{U}_L = \langle U_{L_1}, \dots U_{L_n}\rangle$ is an open neighborhood of L, and there fore $\{\mathfrak{U}_L : L \in \mathfrak{L}\}$ is a covering of $\mathfrak{L}$ by open collections. But $\mathfrak{L}$ is Lindelöf, so there exists a countable subcollection $\{L_1, \dots, L_m, \cdots\}$ of $\mathfrak{L}$ such $\{\mathfrak{U}_{L_1}, \dots, \mathfrak{U}_{L_M}, \cdots\}$ is a covering of $\mathfrak{L}$. Hence, finally, $\{\mathfrak{U}_{L_{i,j}} : i \in N;\ j = 1, \dots, n(L)\}$ is a countable subcollection of $\mathfrak{U}$ which covers A.$\diamondsuit$

Proposition 3.15. *Let X be a noncompact Hausdorff space. Then the space $\mathfrak{K}(X)$ is a LC-space if and only if X is a LC- space.*

Proof. $\Rightarrow$: Let $\mathfrak{K}(X)$ is a LC - space. The subspace $\mathfrak{X} = \{\{x\} : x \in X\} \subset \mathfrak{K}(X)$ is homeomorphic to X. Furthermore, the subspace $\mathfrak{X}$ is closed in $\mathfrak{K}(X)$ and by Proposition 3.8., X is a LC - space.

$\Leftarrow$: Let $\mathfrak{L}$ be a Lindelöf subset of $\mathfrak{K}(X)$. Then, by Lemma 3.14., the set $A = \|\mathfrak{L}\| = \cup\{L \in \mathfrak{L}\}$ is a Lindelöf subset of X. Since the space X is LC, we have that $cl_X(A)$ are compact subset in X. The collection $exp(A) \subset \mathfrak{K}(X)$ and $\mathfrak{L} \subset exp(A)$. Furthermore, the subspace $exp(A)$ is closed and $exp(A) \in \mathfrak{K}(\mathfrak{K}(X))$. Hence, finally $cl_{\mathfrak{K}(X)}(\mathfrak{L}) \in \mathfrak{K}(\mathfrak{K}(X))$.$\diamondsuit$

Definition 3.5. A pair (Y, c), is called a LC extension of space X, if Y is a LC space and $c : X \longrightarrow Y$ is a homeomorphic embedding of X in Y such that $cl_Y(c(X)) = Y$.

Theorem 3.16. *Let X be a Tychonoff space which is not a LC space and let cX be a compactification of X with the following properties:*

(a) The set $\mathfrak{L}_{cX}$ is not empty set.

(b) The set $\mathfrak{L}_{cX} \subset cX \setminus X$.

Then there exists a LC extension ΛX of X such that ΛX is a subspace of cX.

Proof. Consider the subspace $LC(X) = cX \setminus \mathfrak{L}_{cX}$ of cX. Since cX is compact, by Theorem 3.7., the remainder $cX \setminus \mathfrak{L}_{cX}$ is a LC subspace of cX. Let ΛX be the closure of $c(X)$ in $LC(X)$ ($c(X) \approx X$). The subspace

$$\Lambda X = cl_{LC(X)}(c(X)) = cl_{cX}(c(X)) \cap LC(X) = cX \cap LC(X).$$

Hence $\Lambda X = LC(X)$. Furthermore, the mapping $i : c(X) \longrightarrow LC(X)$ defined as $i(y) = y,\ y \in c(X)$, is a homeomorphic embedding of $c(X)$ in $LC(X)$. The mapping $i \circ c : X \longrightarrow LC(X)$ is a homeomorphic embedding of X in $LC(X)$. By Definition 3.5., pair $(LC(X), i \circ c)$ is a LC extension of space X.$\diamondsuit$

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